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BLOCKING s-DIMENSIONAL SUBSPACES BY LINES IN $PG(2s,q)$

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We investigate sets of lines in $PG(2s,q)$ such that every s-dimensional subspace contains a line of this set. We determine the minimum number of lines in such a set and show that there is only one type of such a set with this minimum number of lines.

1. Introduction

The origin of this paper is the following problem. What is the minimum cardinality of a set B of lines of $PG(4,q)$ such that every plane contains at least one line of B?

The first idea might be that the best thing to do is to take the lines of a hyperplane of $PG(4,q)$. If this were true, this would yield a counter-example of the following conjecture of Kahn and Kalai [5] in the special case that $n=q^4+q^3+q^2+q+1$.

CONJECTURE: Any intersecting hypergraph $\mathcal{F}$ on n vertices can be covered by n pairs of vertices.

Here a pair of vertices covers an edge F of $\mathcal{F}$, if both vertices are contained in F. However, taking the lines in a hyperplane is not optimal, since Brouwer [3] noticed that the $q^4+q^3+q^2+q+1$ lines in the orbit of a Singer-cycle of $PG(4,q)$ block all planes, and thus the conjecture is true in the particular case that $\mathcal{F}$ consists of the planes of $PG(4,q)$. We shall see soon that there exist even better examples.

More generally, consider the space $PG(d,q)$ and a set B of lines such that each s-space contains a line of B, where $0<s<d$. We are looking for a set B of minimal cardinality. If $d\leq 2s-1$, this problem is solved by the following theorem of Beutelspacher and Ueberberg [1]:

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Result 1.1. *Let B be a family of lines of $PG(d,q)$ where $d \leq 2s-1$. Suppose that every s-dimensional subspace contains at least one element of B. Then*

$$|B| \geq \frac{q^{2(d+1-s)}-1}{q^2-1},$$

where equality holds iff B is a geometric spread in a $(2d-2s+1)$-dimensional subspace.

Now consider the space $PG(2s,q)$. Consider a point Q and a set F of $(q^{2s}-1)/(q^2-1)$ planes on Q that form a geometric spread in the quotient geometry on Q. Fix an $(s+1)$-space U on Q. Let B consist of the $(q^{s+1}-1)/(q-1)$ lines of U through Q together with the $q^2(q^{2s}-1)/(q^2-1)$ lines that lie in a plane of F but do not contain Q. Then every s-space has a line in B. Since the s-subspaces of $PG(2s,q)$ provide an example of an intersecting family, this shows that the above conjecture is true in this case. In this paper, we show that this example is optimal.

Theorem 1.2. *Suppose that B is a set of lines in $PG(2s,q)$, such that every s-dimensional subspace contains a line of B. Then*

$$|B| \geq \frac{q^{2s+2}-q^2}{q^2-1} + \frac{q^{s+1}-1}{q-1}$$
$$= (q^{2s}+q^{2s-2}+q^{2s-4}+\cdots+q^2)+(q^s+q^{s-1}+q^{s-2}+\cdots+1)$$

and the above example is the only one in which equality holds.

The most general question in this context is probably the following: given $0 \leq t,s < d$, how many t-subspaces of $PG(d,q)$ are needed to block all s-subspaces. The two above results settle the case $t=1$ and $d \leq 2s$. The case $t=0$ was settled by Bose and Burton [2]. In general, the problem seems to be quite difficult.

We shall prove the theorem by induction on s. Here the case $s=1$ is obvious: in this case, B is a set of lines of $PG(2,q)$ such that every line contains a line of B, i.e. B must be the set of lines of $PG(2,q)$, which is the (slightly degenerated) example given above.

From now on we assume that $s \geq 2$ and that the theorem holds for smaller s.

2. A view on $PG(2s-1,q)$

In this section we use the induction hypothesis, applied to $PG(2s-2,q)$, to prove some results on $PG(2s-1,q)$. These results will be used in the proof of Theorem 1.2 considering hyperplanes or quotient spaces.

Lemma 2.1. *Let C be a set of lines in $PG(2s-1,q)$ such that each s-space contains a line of C. Let P be a point contained in exactly k lines of C.*

(a) $|C|\geq \frac{q^{2s}-q^2}{q^2-1}+k$

(b) $k=0 \Longrightarrow |C|\geq \frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}$

Proof. Part (b) follows immediately from the induction hypothesis, because in the quotient geometry on P, whose dimension is $2(s-1)$, the set C becomes a set of lines such that each $(s-1)$-space contains a line of C.

To prove (a), we distinguish two cases.

Case 1: Each point except P lies on a line of C not on P. In this case the number of lines of C not containing P is at least $((q^{2s}-1)/(q-1)-1)/(q+1) > (q^{2s}-q^2)/(q^2-1)$, which implies (a).

Case 2: There exists a point $P' \neq P$ not on a C-line except possibly $\langle P',P\rangle$. Let C' be the set of C-lines not containing P together with the lines through P in an arbitrary s-space through P, but not through P'. Then $|C'|=|C|-k+(q^s-1)/(q-1)$, and in the quotient space on P' each $(s-1)$-space contains a line of C'. So (a) follows from the induction hypothesis. ∎

Lemma 2.2. *Let S be a set of points of $PG(2s-1,q)$ with $|S|=q+1-k$.*

(a) *There are at least kq^{2s-2} hyperplanes not meeting S.*

(b) *If the points of S are not collinear, then there are at least $(k+1)(q^{2s-2}-q^{2s-3})$ hyperplanes not meeting S.*

Proof. Let $P\in S$. Then P is contained in $(q^{2s-1}-1)/(q-1)$ hyperplanes. Each other point of S is contained in exactly q^{2s-2} hyperplanes not containing P. Altogether these are at most $(q^{2s-1}-1)/(q-1)+(q-k)q^{2s-2}=(q^{2s}-1)/(q-1)-kq^{2s-2}$ hyperplanes, i.e. kq^{2s-2} hyperplanes are missing. This proves (a).

If the points of S are not collinear, we can suppose $P_1,P_2,P_3\in S$ to be not collinear. The point P_1 is contained in $(q^{2s-1}-1)/(q-1)$ hyperplanes. The point P_2 is contained in q^{2s-2} hyperplanes not containing P_1. The point P_3 is contained in $q^{2s-2}-q^{2s-3}$ hyperplanes containing neither P_1 nor P_2. Each other point is contained in at most $q^{2s-2}-q^{2s-3}$ planes containing none of P_1,P_2,P_3. Altogether these are at most $(q^{2s-1}-1)/(q-1)+q^{2s-2}+(q-k-1)(q^{2s-2}-q^{2s-3})=(q^{2s}-1)/(q-1)-(k+1)(q^{2s-2}-q^{2s-3})$ hyperplanes, i.e. $(k+1)(q^{2s-2}-q^{2s-3})$ hyperplanes are missing. This proves (b). ∎

Lemma 2.3. *Let L be a line of $PG(2s-1,q)$, and let P be a point of L. Let C be a set of lines such that every $(s-1)$-space containing no point of $L\setminus\{P\}$ contains a line of C.*

(a) $|C|\geq \frac{q^{2s+2}-q^4}{q^2-1}+\frac{q^{s+1}-q}{q-1}$

(b) *If* $|C|<\frac{q^{2s+2}-q^4}{q^2-1}+\frac{q^{s+1}-q}{q-1}+k(q-1)$, *then there are at least* $\frac{q^{s+1}-1}{q-1}-k$ *lines of* C *through* P.

(c) *If* $|C|\leq\frac{q^{2s+2}-q^4}{q^2-1}+\frac{q^{s+1}-1}{q-1}$, *then there are at least* $\frac{q^{s+1}-q}{q-1}$ *lines of* C *through* P.

Proof. Let r be the number of C-lines on P. In the proof of (a) we can assume C to be of minimal cardinality. However, if we replace the C-lines on P by the lines on P in an arbitrary $(s+1)$-space on L, excluding the line L itself, we get another set of lines fulfilling the hypotheses. Hence we can assume $r\leq(q^{s+1}-q)/(q-1)$. Furthermore, we can assume that no line of C meets L outside P.

There are q^{2s-2} hyperplanes on P not containing L. By induction hypothesis each of these hyperplanes contains at least $(q^{2s}-q^2)/(q^2-1)+(q^s-1)/(q-1)$ lines of C. If we count all incident pairs (X,H), where X is a line of C and H a hyperplane on P not containing L, then each C-line through P is counted exactly q^{2s-3} times, whereas the C-lines missing L are counted exactly q^{2s-4} times. This yields

$$rq^{2s-3}+(|C|-r)q^{2s-4}\geq q^{2s-2}\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right),$$

from which follows

$$|C|\geq\frac{q^{2s+2}-q^4}{q^2-1}+q\frac{q^{s+1}-q}{q-1}-r(q-1).$$

From $r\leq(q^{s+1}-q)/(q-1)$ we get (a).

If $r<(q^{s+1}-1)/(q-1)-k$, i.e. $r\leq(q^{s+1}-q)/(q-1)-k$, the cardinality of C grows by another $k(q-1)$, which proves (b).

The assertion (c) follows directly from (b) with $k=1$, except that possibly $q=2$ and there are exactly $(q^{s+1}-1)/(q-1)-2$ lines of C containing P. From now on we assume that we have this situation. Then we have $s\geq3$. (If $s=2$, the set C consists of all lines not containing a point of $L\setminus\{P\}$.)

Since all used inequalities are sharp, we can use the characterization of Theorem 1.2. So for any hyperplane H intersecting L in P the set of C-lines through P in H consists either of two lines or of three coplanar lines or of the set of lines through P in an s-dimensional subspace. The first case is not possible, because otherwise we could find a hyperplane without this property. Hence for two C-lines meeting in P the third coplanar line through P is either L or a C-line. Hence in the quotient space on L the set of C-lines through P must be a subspace of dimension $s-1$ (by cardinality). Thus the set of C-lines through P is exactly the set of lines through P in an $(s+1)$-dimensional subspace on L except the line L itself and a second line. But now we find a hyperplane having a forbidden intersection with C. ∎

Lemma 2.4. *Let L be a line of $PG(2s-1,q)$, and let C be a set of lines such that every $(s-1)$-space containing no point of L contains a line of C. Then $|C| \geq \frac{q^{2s+2}-q^4}{q^2-1}$.*

Proof. Choose a point P of L. Adding to C the lines through P in an arbitrary $(s+1)$-space on L, excluding the line L itself, we get a set C' fulfilling the hypotheses of Lemma 2.3. Applying this lemma, the assertion follows. ∎

3. Proof of the Theorem

Lemma 3.1. *Let B be a set of lines of $PG(2s,q)$ such that each s-space contains a line of B. Let P be a point contained in exactly $q+1-k$ lines of B, where $k \geq 1$.*

(a) $|B| \geq \frac{q^{2s+2}-q^2}{q^2-1} + kq^s - 2k + q + 2$.

(b) *If $|B| \leq \frac{q^{2s+2}-q^2}{q^2-1} + 2q^s - 2q^{s-1} + q - 2$, then $k = 1$, and the q lines of B containing P are coplanar.*

(c) *If $k \leq q$, then $|B| \geq \frac{q^{2s+2}-q^2}{q^2-1} + k\frac{q^{s+1}-1}{q-1} - q^2 - 2k + q + 1$.*

(d) *If $k \leq q$ and $|B| \leq \frac{q^{2s+2}-q^2}{q^2-1} + 2q^s - q^2 + q - 3$, then $k=1$, and the q lines of B containing P are coplanar.*

Proof. According to Lemma 2.2 (a), applied to the quotient space on P, there are at least kq^{2s-2} hyperplanes on P containing no B-line through P. According to Lemma 2.1, each of these hyperplanes contains at least $(q^{2s}-q^2)/(q^2-1) + (q^s-1)/(q-1)$ lines of B, whereas the other $(q^{2s}-1)/(q-1) - kq^{2s-2}$ hyperplanes on P contain at least $(q^{2s}-q^2)/(q^2-1)$ lines of B not containing P.

Counting pairs (L,H), where L is a B-line not containing P and H is a hyperplane on L and P, one gets

$$kq^{2s-2}\left(\frac{q^{2s}-q^2}{q^2-1} + \frac{q^s-1}{q-1}\right) + \left(\frac{q^{2s}-1}{q-1} - kq^{2s-2}\right)\frac{q^{2s}-q^2}{q^2-1} \leq (|B| + k - q - 1)\frac{q^{2s-2}-1}{q-1},$$

which yields

$$|B| + k - q - 1 \geq \frac{q^{2s+2}-q^2}{q^2-1} + k\frac{q^{3s-2}-q^{2s-2}}{q^{2s-2}-1}.$$

Since $(q^{3s-2}-q^{2s-2})/(q^{2s-2}-1) = q^s - 1 + (q^s-1)/(q^{2s-2}-1) > q^s - 1$, we get (a).

In the case of (b), we get $k=1$ from (a). If the q lines of B containing P are not coplanar, Lemma 2.2 (b) yields

$$2(q^{2s-2}-q^{2s-3})\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right)+\left(\frac{q^{2s}-1}{q-1}-2(q^{2s-2}-q^{2s-3})\right)\frac{q^{2s}-q^2}{q^2-1}$$
$$\le (|B|-q)\frac{q^{2s-2}-1}{q-1},$$

which implies that

$$|B|-q\ge \frac{q^{2s+2}-q^2}{q^2-1}+2\left(q^s-q^{s-1}-1+\frac{q^{2s-3}+q^s-q^{s-1}-1}{q^{2s-2}-1}\right).$$

This gives a contradiction. So we have proved (b).

To prove (c) and (d), let L_0 be a B-line containing P, which exists, since now $k\le q$. We count the pairs (L,H), where L is a B-line not intersecting L_0 and H is a hyperplane on L intersecting L_0 in P. This gives the inequality

$$kq^{2s-2}\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right)+\left(q^{2s-1}-kq^{2s-2}\right)\frac{q^{2s}-q^2}{q^2-1}\le (|B|+k-q-1)q^{2s-3},$$

which yields (c).

In the situation of (d), we get $k=1$ from (c). If the q lines of B containing P are not coplanar, Lemma 2.2 (b) yields

$$2(q^{2s-2}-q^{2s-3})\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right)+\left(q^{2s-1}-2(q^{2s-2}-q^{2s-3})\right)\frac{q^{2s}-q^2}{q^2-1}$$
$$\le (|B|-q)q^{2s-3},$$

which implies

$$|B|\ge \frac{q^{2s+2}-q^2}{q^2-1}+2q^s-q^2+q-2,$$

which is a contradiction. So we have proved (d). ∎

From now on let B be a set of lines of $PG(2s,q)$ with $|B|\le (q^{2s+2}-q^2)/(q^2-1)+(q^{s+1}-1)/(q-1)$ such that each s-space contains a line of B.

Lemma 3.2. (a) *The minimal number of B-lines through a point P is q.*

(b) *If there are exactly q lines of B through a point P, these lines are coplanar.*

Proof. If there were at least $q+1$ lines of B through each point, we would have $|B|\ge (q^{2s+1}-1)/(q-1)$, which yields a contradiction. Hence there is a point lying on at most q lines of B.

Lemma 3.1 (b) shows that every point P lies on at least one line of B. Now the assertion follows from Lemma 3.1 (b), except in the case $q=3, s\geq 3$, where it follows from Lemma 3.1 (d), and the case $q=2$, where (a) follows from Lemma 3.1 (a) and (b) is trivial. ∎

A point P contained in exactly q lines of B is called an *affine point*. The plane through P containing the q lines of B through P is denoted by $E(P)$.

Lemma 3.3. *Consider an affine point P and the B-lines $H_1,\dots,H_q$ through P. Let $E:=E(P)$, and let H_0 be the line through P in E that is not in B.*

(a) *The number of B-lines containing a point of $E\setminus H_0$ is either q^2 or q^2+1.*

(b) *Each B-line L which contains an affine point contains only affine points, except that possibly one point of L lies on $q+1$ lines of B.*

(c) *There are at least q^2 lines of B contained in E. There is at most one B-line outside E that intersects E in a point outside H_0.*

(d) *There is exactly one point Q on E which lies on more than $q+1$ lines of B.*

(e) *If $P'\in E$ is affine, then $E(P')=E$.*

(f) *All q^2 lines of E missing Q are elements of B. If there is no other B-line having a point of $E\setminus\{Q\}$, then every point of $E\setminus\{Q\}$ is affine. Otherwise there is a unique other B-line G having a point of $E\setminus\{Q\}$, and then the set of affine points of $E\setminus\{Q\}$ is just the set of points outside G.*

(g) *There are at least $\frac{q^{s+1}-q}{q-1}$ lines of B through Q that are not contained in E.*

Proof. Let B' be the set of B-lines containing a point of $E\setminus H_0$, i.e. the set of B-lines meeting (at least) one of $H_1,\dots,H_q$ (including the H_i themselves).

Applying Lemma 2.3 (a) to the quotient space on P (where L corresponds to E and P corresponds to H_0), we see that there are at least $(q^{2s+2}-q^4)/(q^2-1)+(q^{s+1}-q)/(q-1)$ lines of B meeting none of $H_1,\dots,H_q$. Hence there remain at most q^2+1 lines of B meeting one of $H_1,\dots,H_q$ (including the H_i themselves), i.e. $|B'|\leq q^2+1$.

Since each point of H_1 is contained in at least q lines of B, there are at least $(q+1)(q-1)=q^2-1$ lines of B meeting H_1 in a point. Hence $|B'|\geq q^2$. This proves (a).

If $|B'|=q^2$, then all points of H_1 are affine. If $|B'|=q^2+1$, there may be one non-affine point of H_1, which lies on exactly $q+1$ lines of B'. This proves (b).

We have shown that q^2 lines of B' must intersect H_1. The same holds for $H_2,\dots,H_q$. If $|B'|=q^2$, this means that all lines of B' are contained in E. (Note that there are no B-lines through P outside E.) So, assertion (c) holds in the case $|B'|=q^2$.

Suppose now that $|B'|=q^2+1$ and assertion (c) fails, i.e. there are two lines L_1, L_2 of B' not contained in E. Since q^2 lines of B' must intersect H_1, at most one line of B' does not intersect H_1. The same holds for $H_2, \dots, H_q$. The line L_1 intersects only one H_i, say H_1. All other lines of B', in particular L_2, must intersect each of $H_2, \dots, H_q$. If $q \geq 3$, this yields a contradiction.

If $q=2$, the line L_2 must intersect H_2 in a point P', and all lines of B' except L_1, L_2 are contained in E. We know that there is at most one B-line not contained in $E=E(P)$ that intersects H_2. Similarly there is at most one B-line not contained in $E(P')$ that intersects H_2. (Note that all points of H_2, including P', are affine.) Since $E(P')=\langle L_2, H_2\rangle \neq E(P)$, there are at most two B-lines intersecting H_2, i.e. one point on H_2 is contained in at most one B-line, which is a contradiction. This proves (c).

Let $P' \in H_1$ be affine. According to (c), any plane containing three B-lines intersecting H_1 must be equal to $E(P')$. Since E contains q^2 lines of B, all meeting H_i, we have $E(P')=E$. This proves (e) for all $P' \in E \setminus H_0$.

As a consequence of (b), all points of E which lie on more than $q+1$ lines of B must be on H_0. Take an affine point P' on $H_1 \setminus \{P\}$ (existing by (b)). As we have just seen, $E(P')=E$. Let H_0' be the unique line through P' in E which is not in B. Then all points of E which lie on more than $q+1$ lines of B must be on H_0'. Thus $Q:=H_0 \cap H_0'$ is the only possible point of E lying on more than $q+1$ lines of B. Next we show that Q really lies on more than $q+1$ lines of B. To do this, we distinguish the cases $s=2$ and $s \geq 3$.

If $s=2$, each of the q^2+q+1 planes on H_0' contains a B-line. This line intersects H_0'. Except of E, at most one of these planes contains a B-line meeting H_0' in a point different from Q. (This is a consequence of (c).) The other q^2+q-1 planes contain distinct B-lines containing Q. Hence there are at least $q^2+q-1 > q+1$ lines of B through Q.

If $s \geq 3$, we apply Lemma 2.3 (c) to the quotient space on P as in the beginning of the proof. We see that there are at least $(q^{s+1}-q)/(q-1) > q(q+1)$ lines of B meeting H_0 outside P. Hence there must be a point of H_0 which is contained in more than $q+1$ lines of B, namely Q. So we have shown (d).

The missing part of (e) now follows by symmetry, using an affine point $P' \in H_1 \setminus \{P\}$. (Note that $\langle P', Q\rangle \notin B$ because of (b); hence all points of $H_0 \setminus \{Q\}$ lie on B-lines through P'.)

Suppose there is a line L of E missing Q which is not in B. By (c), one of $L \cap H_1, L \cap H_2$ must be affine, say P'. Then $L, \langle P', Q\rangle \notin B$, which is a contradiction to (e). So we have shown that all lines of E missing Q are elements of B.

If there is a B-line G through Q in E, then clearly all points of G are non-affine. If there is another non-affine point R of E, then by (b) all points of E not on $\langle R, Q\rangle$ must be non-affine, and again applying (b) we see that no point of E can be affine, which is a contradiction. Hence all points of E outside G must be affine, and so no further B-line can intersect E outside Q. That is, (f) holds when there is a B-line through Q in E.

If there is no B-line through Q in E, then the non-affine points of $E\setminus\{Q\}$ are just the intersection points of E with B-lines not contained in E. By (b) and (c) we see that there is at most one such line. Thus (f) holds.

Take a line L through Q in E such that all points of $L\setminus\{Q\}$ are affine (by (f)). Let P' be a point of $L\setminus\{Q\}$. Now apply Lemma 2.3 (c) to the quotient space on P' as in the proof of (d). Since all B-lines meeting L that are not contained in E must meet L in Q, we get (g). ∎

We now fix a point Q that lies on at least $(q^{s+1}-q)/(q-1)$ lines of B. Let $\bar{B}$ be the set of B-lines not containing Q. A point is called *quasiaffine* either if it is affine or if it is contained in exactly $q+1$ lines of B one of which contains Q.

Lemma 3.4. (a) *A quasiaffine point lies on exactly q lines of $\bar{B}$. A non-quasiaffine point different from Q lies on more than q lines of $\bar{B}$.*

(b) *The point Q is the only point that lies on at least $(q^{s+1}-q)/(q-1)$ lines of B.*

(c) *There are at most $q+1$ points different from Q which are not quasiaffine.*

Proof. By Lemma 3.3 (b) we see that an affine point doesn't lie on a B-line through Q. This implies (a).

Since Q lies on at least $(q^{s+1}-q)/(q-1)$ lines of B, we have $|\bar{B}|\le|B|-(q^{s+1}-q)/(q-1)\le(q^{2s+2}-q^2)/(q^2-1)+1$. If l_i is the number of points different from Q lying on exactly i lines of $\bar{B}$, we have $\sum_{i\ge q} il_i=|\bar{B}|(q+1)$. Since $\sum l_i=(q^{2s+1}-q)/(q-1)$, this implies

$$\sum_{i>q}(i-q)l_i=|\bar{B}|(q+1)-(q^{2s+2}-q^2)/(q-1)\le q+1.$$

This implies (c), and shows furthermore that a point $\neq Q$ lies on at most $2q+2$ lines of B. Assume that there exists a second point Q' on at least $(q^{s+1}-q)/(q-1)$ lines of B. Then $(q^{s+1}-q)/(q-1)\le 2(q+1)$, which implies that $q=s=2$ and we have equality. Thus Q and Q' lie on $2q+2=6$ lines of B, the line $H:=\langle Q,Q'\rangle$ is in B, and every other point is quasiaffine. Hence, the set M of points that lie on exactly $q+1$ lines of B are just the points other than Q,Q' on the B-lines on Q, so $|M|=11$. Similarly, the points of M are the points other than Q,Q' on the B-lines on Q'. Thus, if E is a plane on H, then either no or all four points of $E\setminus H$ are in M. This implies that $E\setminus H$ has either no or four points in M. Since M has a unique point on H, it follows that $|M|-1$ is divisible by four, a contradiction. This proves (b). ∎

Lemma 3.5. *Let P be a quasiaffine point.*

(a) *The B-lines through P are contained in a plane $E := E(P)$ through Q.*

(b) *The number of $\bar{B}$-lines meeting E is either q^2 or q^2+1.*

(c) *Each B-line L which contains a quasiaffine point contains only quasiaffine points, except that possibly one point lies on $q+1$ lines of B missing Q, and except that possibly Q lies on L.*

(d) *All lines in E missing Q are contained in B. There is at most one B-line intersecting E in a point different from Q.*

(e) *All points of $E\setminus\{Q\}$ with one possible exception are quasiaffine. If $P' \in E$ is quasiaffine, then $E(P') = E$.*

Proof. If P is affine, all the assertions have been proved above. So we may assume that P lies on $q+1$ lines of B one of which contains Q.

Suppose the $q+1$ lines of B through P are not coplanar. Then by Lemma 2.2 (b), applied to the quotient space on P, there are at least $q^{2s-2} - q^{2s-3}$ hyperplanes containing no B-line through P, and thus not containing Q. According to Lemma 2.1, each of these hyperplanes contains at least $(q^{2s}-q^2)/(q^2-1)+(q^s-1)/(q-1)$ lines of B, whereas the other $q^{2s-1}-q^{2s-2}+q^{2s-3}$ hyperplanes on P not containing Q contain at least $(q^{2s}-q^2)/(q^2-1)$ lines of B not containing P.

Counting pairs (L,H), where L is a B-line not containing P and H is a hyperplane on L and P missing Q, one gets

$$(q^{2s-2}-q^{2s-3})\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right)+(q^{2s-1}-q^{2s-2}+q^{2s-3})\frac{q^{2s}-q^2}{q^2-1} \le |\bar{B}|q^{2s-3}.$$

Since $|B| \le (q^{2s+2}-q^2)/(q^2-1)+(q^{s+1}-1)/(q-1)$ and since Q is by Lemma 3.3 (g) on at least $(q^{s+1}-q)/(q-1)$ lines of B, we have $|\bar{B}| \le \frac{q^{2s+2}-q^2}{q^2-1}+1-q$. This yields a contradiction, except in the case $q=2, s=2$, when equality holds.

In order to complete the proof of part (a), assume that $q=s=2$ and that the three B-lines $L_0 := PQ$, L_1 and L_2 on P do not lie in a plane. Then $|B|=27$ and Q lies on exactly $(q^{s+1}-q)/(q-1)=6$ lines of B. Put $L_0=\{P,Q,P'\}$. Then P' lies on at least three B-lines, and thus there are at least ten B-lines that meet L_0. There are exactly 17 planes on P that do not contain a B-line on P. Since each of these planes contains a B-line, it follows that there are exactly 17 lines in B that are skew to L_0, L_1 and L_2, and that the other ten B-lines meet L_0. Furthermore, P' lies on exactly three B-lines. The two points of $L_1\setminus\{P\}$ lie each on at least three B-lines and these meet L_0, since they are not skew to L_1. Thus the three B-lines on P' lie in the plane $\langle L_0, L_1\rangle$. But similarly, they lie in the plane $\langle L_0, L_2\rangle$. Since these two planes are distinct, this is a contradiction. This proves (a).

From Lemma 2.4, applied to the quotient space on P, we see that there are at least $(q^{2s+2}-q^4)/(q^2-1)$ lines of B skew to E. Together with Lemma 3.3 (g) we see that the number of B-lines meeting E but not Q is at most q^2+1. Since each point lies on at least q lines of $\bar{B}$, we get (b). The assertions (c), (d) are proved as Lemma 3.3 (b), (c), regarding only lines of $\bar{B}$, and replacing "affine" by "quasiaffine". (For (d) note that there are only q^2 lines in E missing Q.) Assertion (e) is a direct consequence of (d). ∎

Lemma 3.6. (a) *The planes $E(P)$, where P is quasiaffine, form a spread in the quotient space on Q.*

(b) *The lines of $\bar{B}$ are just the lines on the planes $E(P)$ which do not contain Q, except possibly one more line G.*

(c) *The non-quasiaffine points are Q and the points of G.*

Proof. From Lemma 3.5 (e) we see that the planes $E(P)$ have only Q as common point. If there are r such planes, they cover $1+r(q^2+q)$ points. If $r \le (q^{2s}-1)/(q^2-1)-1$, then there remain q^2+q points uncovered, in contradiction to Lemma 3.4 (c). So $r \ge (q^{2s}-1)/(q^2-1)$, i.e. the planes cover all points. This proves (a).

From part (a) and Lemma 3.3 (g) we see that there is at most one B-line G which is neither on Q nor in a plane $E(P)$. This proves (b). Assertion (c) is a consequence of (b). ∎

Lemma 3.7. *Consider the quotient space on Q. Let C be the set of points corresponding to the B-lines through Q.*

(a) *If a line contains three points of C, then L is contained in C.*

(b) *If two intersecting lines both contain two points of C, then their point of intersection is contained in C.*

(c) *The set C is the set of points of an s-space.*

Proof. Let T be the set consisting of the B-lines on Q and the B-lines that do not lie in a plane $E(P)$. Then $|T| \le (q^{s+1}-1)/(q-1)$. By Lemma 3.3 (g), at most one line of T does not contain Q.

Consider a point in the quotient space on Q which is not in C. This point corresponds to a line L through Q that is not in B. Let P be an affine point on this line (existing by Lemma 3.6 (c)). Now we can use the argument of the proof of Lemma 3.3 (g) (applying Lemma 2.3 (c)) to see that except $E(P)$ there are at least $(q^{s+1}-1)/(q-1)-1$ planes on L which contain a B-line. These B-lines lie in T. Since $|T| \le (q^{s+1}-1)/(q-1)$, it follows that there is no plane on L that contains more than two lines of T, and there is at most one plane on L that contains two lines of T. (For the plane $E(P)$ the latter is a consequence of Lemma 3.3 (f).) This proves (a) and (b). We have also seen that if a plane on Q contains a line H on Q that is not in B as well as two B-lines on Q, then $|T| = (q^{s+1}-1)/(q-1)$ and all lines of T meet H.

By (b), the set C fulfills the Veblen-Young Axiom. Hence it is a generalized projective space, i.e. the disjoint union of projective spaces (see [4], Theorem 2.5). Because of (a), these projective spaces are linear subspaces. Furthermore we see from the disjointness that the sum of the dimensions of these subspaces is at most $2(s-1)$. Since the number of points of C is either $(q^{s+1}-1)/(q-1)-1$ or $(q^{s+1}-1)/(q-1)$, we get (c), except in the case $q=2$, when C can be the disjoint union of two $(s-1)$-dimensional subspaces.

Assume that the latter case occurs. Then $q=2$ and $PG(2s,q)$ has two s-dimensional subspaces U_1 and U_2 with $U_1 \cap U_2 = \{Q\}$ such that the B-lines on Q are the lines on Q that are in U_1 or U_2. Furthermore Q lies on exactly $(q^s-q)/(q-1)$ lines of B. For lines L_i of U_i on Q, the plane $\langle L_1, L_2\rangle$ contains a third line H on Q, which is not in B. We have seen above that this implies that $|T|=(q^{s+1}-1)/(q-1)$ and that all lines of T meet H. Hence there exists exactly one line $G \in T$ that does not contain Q, and this line meets H. Part (a) shows that different pairs (L_1, L_2) of lines L_i yield different lines H. Since U_i contains q^s-1 lines on Q, this gives $(q^s-1)^2 \geq 3^2$ different lines H on Q that meet G. But G has only three points, a contradiction. ∎

Now we can prove Theorem 1.2. From Lemma 3.7 (c) we see that the set of B-lines through Q is exactly the set of lines through Q in an $(s+1)$-space on Q. From Lemma 3.6 we see that there is a set of planes F forming a spread in the quotient space on Q such that the lines of the planes of F which do not contain Q are B-lines. By the cardinality of B, these are all lines of B. So we need only show that the spread F is geometric. This follows from Result 1.1, since each s-space not containing Q contains a line of B and hence must intersect a plane of F in a line, which means in the quotient space on Q that each s-space contains a line of F. ∎

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